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The Supersymmetry Approaches to the Non-central Kratzer Plus Ring-Shaped Potential

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Abstract In this paper, we study the Schrödinger equation with non-central modified Kratzer potential plus a ring-shaped like potential, which is not spherically symmetric. We connect the corresponding Schrödinger equation to the Laguerre and Jacobi equations. These lead us to have some raising and lowering operators which are first order equations. We take advantage from these first order equations and discuss the supersymmetry algebra. And also we obtain the corresponding partner Hamiltonian for Kratzer potential and investigate the commutation relation for the generators algebra.

Keywords Modified Kratzer potential · Schrödinger equation · Supersymmetry approaches · Raising and lowering operators

1 Introduction

One of the important tasks of quantum mechanics is to find exact solutions of the Schrödinger equation for certain potentials of physical interest. It is well known that the exact solutions of this equation are only possible for certain potentials such as the Coulomb and Harmonic oscillator potential. In recent years, considerable efforts have been made to obtain the exact solution of the ring-shaped potentials [1]. In particular, the Coulombic ring-shaped potential [2] revived in quantum chemistry by Hartmann [3] and the oscillatory ring-shaped

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potential, systematically studied by Quesne [4], have been investigated from a quantum mechanical viewpoint by using various approaches. The Kratzer potential [5, 6] plays an important role in atomic and molecular physics and quantum chemistry. Recently, Berkdemir et al. [7] proposed a new potential which is called the modified Kratzer's type of molecular potential $V(r) = D_e[\frac{(r-r_e)}{r}]^2$, where D_e is the dissociation energy and r_e is the equilibrium internuclear separation. Also recently, Chen and Dong [8] found a new ring-shaped potential and obtained the exact solution of the Schrödinger equation for the Coulomb potential add this new ring-shaped potential. The purpose of the paper is to solve the Schrödinger equation for the modified Krazter potential plus this new ring-shaped potential. This ring-shaped potential is a possible application to ring-shaped organic molecules like cyclic polyenes and benzene. In spherical coordinates, this potential is defined as

$$v(r,\theta) = D_e\left(\frac{r-r_e}{r}\right)^2 + \left[\frac{\beta' \cos^2\theta}{r^2\sin^2\theta}\right], \tag{1}$$

where β' is positive real constant. We find that the potential (1) reduces to the modified Krazter potential in the limiting case of $\beta' = 0$. There are different methods used to obtain the exact solutions of the Schrödinger equation for the ring-shaped potential. In the other hand supersymmetry in quantum mechanics is based on the concept of factorization in the context of shape invariant quantum mechanical problems [9–13]. If a quantum mechanical problem possesses supersymmetry, we can then factorize the Hamiltonian of the system in terms of a product of first order differential operators leading to shape invariant equations. In this approach, the Hamiltonian is decomposed once in terms of the product of raising and lowering operators and once again as the product of lowering and raising operators, in such a way that the corresponding quantum states of successive levels, are their eigenstates. These Hamiltonians are called supersymmetric partner of each other. In fact, the three separate topics, the factorization method, supersymmetry in quantum mechanics and shape invariance, nowadays have converged to each other. All these give us motivation to study supersymmetry approaches for the corresponding potential.

2 Seperating Variables of the Schrödinger Equation

Now we are going to consider the corresponding potential in spherical coordinates. In order to separate the radial and angular part we consider the following expression,

$$\Psi(r,\theta,\phi) = R(r)H(\theta)\Phi(\varphi). \tag{2}$$

The substituting this equation to the general form of corresponding Schrödinger, we have radial and angular part of equations as following,

$$\left[\frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial}{\partial r}\right) - \frac{L_{D-1}^2}{r^2}\right]R(r) + \frac{2\mu}{\hbar^2}\left[E - D_e\left(\frac{r-r_e}{r}\right)^2\right]R(r) = 0, \tag{3}$$

and

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{m'^2}{\sin^2\theta} - \frac{2\mu\beta'}{\hbar^2}\frac{\cos^2\theta}{\sin^2\theta} + l(l+D-2)\right]H(\theta) = 0, \tag{4}$$

$$\frac{\partial^2\Phi}{\partial\varphi^2} + m'^2\Phi(\varphi) = 0, \tag{5}$$

where

$$L_{D-1}^2 = l(l+D-2). \tag{6}$$

The solutions in (5) is periodic and must satisfy the period boundary condition

$$\Phi(2\pi+\varphi) = \Phi(\varphi)$$

from which we obtain:

$$\Phi_{m'}(\varphi) = \frac{1}{\sqrt{2\pi}} \exp(\pm i m' \varphi), \quad m' = 1, 2, \ldots, k. \tag{7}$$

Next step we try to solve (3) and (4), so the radial part is (3):

$$\frac{d^2 R(r)}{dr^2} + \left[\frac{2\mu}{\hbar^2}(E - D_e) + \frac{4D_e}{r} - \frac{Z + (\frac{2\mu D_e r_e^2}{\hbar^2})}{r^2}\right] R(r) = 0, \tag{8}$$

where

$$Z = \frac{1}{4}(M-1)(M-3), \tag{9}$$

and

$$Z = D + 2\lambda. \tag{10}$$

On the other hand (4) represents the angular part of equation, so we have:

$$\frac{d^2 H(\theta)}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{dH(\theta)}{d\theta} + \left[l(l+D-2) - \frac{m'^2 + \frac{2\mu\beta'}{\hbar^2}\cos^2\theta}{\sin^2\theta}\right] H(\theta) = 0. \tag{11}$$

Therefore, we will try to solve (8) and (11) by following section.

3 The Solutions of the Radial Part with Factorization Method

In order to solve the radial part of (8) we consider the following variables,

$$R(r) = g(r) r^{-\frac{(D-1)}{2}}, \tag{12}$$

$$g(r) = G(r) L(r), \tag{13}$$

and compare the following associated Laguerre equation [14–17],

$$r L''_{n,m}(r) + [1 + \alpha - \beta r] L'_{n,m}(r) + \left[\left(n - \frac{m}{2}\right)\beta - \frac{m}{2}\left(\alpha + \frac{m}{2}\right)\frac{1}{r}\right] L_{n,m}(r) = 0, \tag{14}$$

the energy spectrum and radial part of wave function are respectively

$$E = D_e - \frac{\hbar^2}{2\mu}\frac{r\beta^2}{4}, \tag{15}$$

$$R(r) = r^{\frac{(\frac{5}{2} + \alpha - D)}{2}} e^{-\frac{\beta r}{2}} L_{n,m}^{\alpha,\beta}(r). \tag{16}$$

Now we are going to the factorize the second order equation from radial part [14–19]. In that case the first order operators A_+ and A_- with respect to m are,

$$A_+(m;r) = \sqrt{r}\frac{d}{dr} - \frac{m-1}{2\sqrt{r}},$$
$$A_-(m;r) = -\sqrt{r}\frac{d}{dr} - \frac{2(\frac{3}{2}+\alpha-D)+m-2}{2\sqrt{r}}, \tag{17}$$

and with respect to n and m are,

$$A_+(n,m;r) = r\frac{d}{dr} - \beta r + \frac{1}{2}(2n+3+2\alpha-2D-m),$$
$$A_-(n,m;r) = r\frac{d}{dr} + \frac{1}{2}(2n-m). \tag{18}$$

4 The Solutions of the Angular Part of Equation

In order to apply factorization method, we introduce a new variable

$$x = \cos\theta,$$

so the angular part of equation is,

$$(1-x^2)H''(x) - 2xH'(x) + \left[Q - \frac{m'^2}{1-x^2} - N\frac{x^2}{1-x^2}\right]H(x) = 0, \tag{19}$$

where

$$Q = \lambda(\lambda+D-2),$$

and

$$N = \frac{2\mu\beta'}{\hbar^2}.$$

We assume the $H(x)$ as follows,

$$H(x) = f(x)p(x), \tag{20}$$

and compare with the following Jacobi equation [13–15]

$$(1-x^2)P''^{(\alpha,\beta)}_{n,m}(x) - [\alpha-\beta+(\alpha+\beta+2)x]P'^{(\alpha,\beta)}_{n,m}(x)$$
$$+\left[n(\alpha+\beta+n+1) - \frac{m(\alpha+\beta+m+(\alpha-\beta)x)}{1-x^2}\right]p^{(\alpha,\beta)}_{n,m}(x) = 0, \tag{21}$$

one can obtain the angular part of wave function,

$$H(x) = \left(\frac{1+x}{1-x}\right)^{(\frac{\beta-\alpha}{4})}(1-x^2)^{\frac{(\alpha+\beta)}{4}}p^{(\alpha,\beta)}_{n,m}(x), \tag{22}$$

and some following results,

$$
\begin{aligned}
n(\alpha+\beta+n+1) &= Q - \frac{(\alpha+\beta)}{2},\\
(\alpha+\beta)^2 &= 0,\\
(\alpha+\beta) &= -2m.
\end{aligned}
\tag{23}
$$

In that case the first order operators corresponding to the angular part of equation with respect to m are,

$$
\begin{aligned}
A_+(m;x) &= \sqrt{1-x^2}\frac{d}{dx} + \frac{m-1}{\sqrt{1-x^2}}x,\\
A_-(m;x) &= -\sqrt{1-x^2}\frac{d}{dx} + \frac{(\alpha-\beta)+(\alpha+\beta+m)x}{\sqrt{1-x^2}},
\end{aligned}
\tag{24}
$$

with respect to n and m are,

$$
\begin{aligned}
A(n,m;x) &= -\sqrt{(1-x^2)}\frac{d}{dx} - nx + \frac{(\alpha-\beta)(n-m)}{\alpha+\beta+2n},\\
A^+(n,m;x) &= \sqrt{(1-x^2)}\frac{d}{dx} + (\alpha+\beta+n)x - \frac{(\alpha-\beta)(\alpha+\beta+n+m)}{\alpha+\beta+2n}.
\end{aligned}
\tag{25}
$$

5 The Supersymmetry Approaches for the Kratzer Potential

In order to discuss the supersymmetry for this model, we have to consider the ground state wave function. By using the Hamiltonian process and ground state wave function, we can obtain the $V_1(r)$ as following solutions of the radial wave equation:

$$
H_1\Psi_0(r) = -\frac{\hbar^2}{2m}\frac{d^2\Psi_0}{dr^2} + V_1(r)\Psi_0(r) = 0,
\tag{26}
$$

$$
V_1(r) = \frac{\hbar^2}{2m}\frac{d^2\Psi_0''(r)}{\Psi_0(r)}.
\tag{27}
$$

Now, we factorize the corresponding Hamiltonian in terms of first order equation, which are called A and A^+

$$
H_1 = A^+A.
\tag{28}
$$

From information of Laguerre equation the first order operators are given by the following equation

$$
\begin{aligned}
A(n,m;r) &= -r\frac{d}{dr} + \frac{1}{2}(2n-m),\\
A^+(n,m;r) &= r\frac{d}{dr} - \beta r + \frac{1}{2}K,
\end{aligned}
\tag{29}
$$

where

$$
K = (2n+3+2\alpha-2D-m).
\tag{30}
$$

So, Hamiltonian H_1 is,

$$H_1 = A^+A = -r^2\frac{d^2}{dr^2} + \left[\beta r - \left(1 + \frac{1}{2}K\right)\right] r\frac{d}{dr} - \frac{1}{2}(2n-m)\beta r + \frac{1}{4}K(2n-m). \quad (31)$$

This lead us the following potential,

$$V_1(r) = W^2(r) - \frac{\hbar}{\sqrt{2m}}W'(r), \quad (32)$$

this equation is known as Ricati equation, where W_r is supper potential,

$$W(r) = -\beta r + \frac{1}{2}K. \quad (33)$$

Finally we have,

$$V_1(r) = \beta^2 r^2 - \beta K r + \frac{1}{4}K^2 + \frac{\hbar}{\sqrt{2m}}\beta. \quad (34)$$

Now, we are going to obtain the Hamiltonian H_2, as $H_2 = AA^+$ which is partner of H_1,

$$\begin{aligned} H_2 &= -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + V_2(r), \\ V_2(r) &= W^2(r) + \frac{\hbar}{\sqrt{2m}}W'(r). \end{aligned} \quad (35)$$

Also the corresponding potential V_2 can be obtain by the following expression,

$$V_2(r) = \beta^2 r^2 - \beta K r + \frac{1}{4}K^2 - \frac{\hbar}{\sqrt{2m}}\beta, \quad (36)$$

where H_2 is,

$$H_2 = -r^2\frac{d^2}{dr^2} - \left[1 - \frac{1}{2}(2n-m)\right] r\frac{d}{dr} + \left[1 - \frac{1}{2}(2n-m)\right]\beta r + \frac{1}{4}K(2n-m). \quad (37)$$

The potential V_1 and V_2 are supersymmetry partner to each other. On the other hand, we have the matrix supersymmetry for The Hamiltonian H_1 and H_2,

$$H = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}, \quad (38)$$

we note here the H_1 and H_2 can make closed algebra and they relate to Bosonic and fermionic operators,

$$Q = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -r\frac{d}{dr} + \frac{1}{2}(2n-m) & 0 \end{bmatrix}, \quad (39)$$

and

$$Q^+ = \begin{bmatrix} 0 & A^+ \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & r\frac{d}{dr} - \beta r + \frac{1}{2}K \\ 0 & 0 \end{bmatrix}. \quad (40)$$

Here, we have the following commutation relation,

$$[H, Q] = [H, Q^+] = 0,$$
$$\{Q, Q^+\} = H, \qquad \{Q, Q\} = \{Q^+, Q^+\} = 0,$$

and

$$[H, Q] = \begin{bmatrix} 0 & 0 \\ H_2A - AH_1 & 0 \end{bmatrix}. \tag{41}$$

In order to satisfy the (41) and

$$H_2A = AH_1, \tag{42}$$

we have β as follows,

$$\beta = \frac{2n + 2\alpha - 2D - m + 7}{2r}, \tag{43}$$

this value of β grantee the (41), $[H, Q] = 0$. Also from (41) we will obtain following expression, $[H, Q^+] = 0$

$$[H, Q^+] = \begin{bmatrix} 0 & H_1A^+ - A^+H_2 \\ 0 & 0 \end{bmatrix} = 0$$
$$\Rightarrow \quad H_1A^+ = A^+H_2. \tag{44}$$

By comparing the left and right hand side of above equation we have two conditions, $2n - m = 4$ and $2n - m = 0$, we have,

$$[H, Q^+] = 0.$$

This completely satisfy the following anti-commutation relations, $\{Q, Q^+\} = H, \{Q, Q\} = \{Q^+, Q^+\} = 0$,

$$\begin{aligned} \{Q^+, Q^+\} = 0 \quad &\Rightarrow \quad \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix} = 0, \\ \{Q, Q\} = 0 \quad &\Rightarrow \quad \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} = 0, \end{aligned} \tag{45}$$

and

$$\begin{aligned} \{Q, Q^+\} = H \quad &\Rightarrow \quad \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}\begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \\ &= \begin{pmatrix} A^+A & 0 \\ 0 & AA^+ \end{pmatrix} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} = H. \end{aligned} \tag{46}$$

We note here the supercharges have a commutation to each other and also we have some degeneracy. Now we are going to continue this process for the angular part,

$$\begin{aligned} A(n, m; x) &= -\surd(1 - x^2)\frac{d}{dx} - nx + \frac{(\alpha - \beta)(n - m)}{\alpha + \beta + 2n}, \\ A^+(n, m; x) &= \surd(1 - x^2)\frac{d}{dx} + (\alpha + \beta + n)x - \frac{(\alpha - \beta)(\alpha + \beta + n + m)}{\alpha + \beta + 2n}, \end{aligned} \tag{47}$$

we take advantage from change of variable

$$x = \cos\theta$$

the first order operators in terms of θ will be as,

$$\begin{aligned} A(n,m;\theta) &= -\frac{d}{d\theta} - n\cos\theta + \frac{(\alpha-\beta)(n-m)}{\alpha+\beta+2n}, \\ A^+(n,m;\theta) &= \frac{d}{d\theta} + (\alpha+\beta+n)\cos\theta - \frac{(\alpha-\beta)(\alpha+\beta+n+m)}{\alpha+\beta+2n}, \end{aligned} \tag{48}$$

where

$$\begin{aligned} \epsilon &= \frac{(\alpha-\beta)(n-m)}{\alpha+\beta+2n}, \\ \gamma &= (\alpha+\beta+n), \\ \rho &= \frac{(\alpha-\beta)(\alpha+\beta+n+m)}{\alpha+\beta+2n}. \end{aligned} \tag{49}$$

The corresponding Hamiltonian for H_1 is,

$$H_1 = -\frac{d^2}{d\theta^2} + n\sin\theta + (\gamma\cos\theta + \rho)\frac{d}{d\theta} + n\gamma\cos^2\theta + m(\alpha-\beta)\cos\theta + -\epsilon\rho, \tag{50}$$

the superpotential for the ground state wave function $W(r) = -\gamma\cos\theta - \rho$ and the potential V_1. Will be obtained by the following expression,

$$\begin{aligned} V_1(\theta) &= (-\gamma\cos\theta - \rho)^2 + \frac{\hbar}{\sqrt{2m}}\gamma\sin\theta \\ &= (\gamma\cos\theta)^2 + \rho^2 + 2\gamma\rho\cos\theta - \frac{\hbar}{\sqrt{2m}}\gamma\sin\theta. \end{aligned} \tag{51}$$

Next step we want to make the partner Hamiltonian $H_2 = AA^+$. The Hamiltonian correspond to V_2 potential is,

$$\begin{aligned} H_2 &= -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + V_2(r), \\ V_2(r) &= W^2(r) + \frac{\hbar}{\sqrt{2m}}W'(r), \end{aligned} \tag{52}$$

the final result for the $V_2(r)$ and corresponding Hamiltonian are respectively,

$$V_2(\theta) = (\gamma\cos\theta)^2 + \rho^2 + 2\gamma\rho\cos\theta + \frac{\hbar}{\sqrt{2m}}\gamma\sin\theta, \tag{53}$$

and

$$H_2 = -\frac{d^2}{d\theta^2} - \gamma\sin\theta + (\epsilon - n\cos\theta)\frac{d}{d\theta} + n\gamma\cos^2\theta + (n\rho - \epsilon\gamma)\cos\theta - \epsilon\rho, \tag{54}$$

where $V_1(\theta)$ and $V_2(\theta)$ are partner to each other, the matrix form for Hamiltonian is,

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}. \tag{55}$$

This Hamiltonian make closed algebra where supercharges Q and Q^+ are respectively,

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\frac{d}{d\theta} - n\cos\theta + \epsilon & 0 \end{pmatrix}, \tag{56}$$

and

$$Q^+ = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{d}{d\theta} - \gamma\cos\theta - \rho \\ 0 & 0 \end{pmatrix}, \tag{57}$$

This also closed form of $sl(1,1)$ algebra, so we have

$$\begin{aligned} &[H, Q] = [H, Q^+] = 0, \\ &\{Q, Q^+\} = H, \{Q, Q\} = \{Q^+, Q^+\} = 0, \end{aligned} \tag{58}$$

and

$$[H, Q] = \begin{bmatrix} 0 & 0 \\ H_2 A - A H_1 & 0 \end{bmatrix}. \tag{59}$$

From (59) we have obtain the following condition,

$$H_2 A = A H_1. \tag{60}$$

In that case if the relation $[H, Q]$ want to be satisfy we will have,

$$\gamma\cos\theta = -\rho. \tag{61}$$

Again we consider the following relation and

$$[H, Q^+] = \begin{bmatrix} 0 & H_1 A^+ - A^+ H_2 \\ 0 & 0 \end{bmatrix} = 0 \quad \Rightarrow \quad H_1 A^+ = A^+ H_2. \tag{62}$$

Also with comparing the left and right hand side of above equation, we obtain,

$$\begin{aligned} (\rho + \epsilon)\gamma \sin\theta &= 0, \\ \alpha - \beta &= 0, \end{aligned} \tag{63}$$

if we apply the $2n - m = 0$, we have,

$$[H, Q^+] = 0.$$

This completely satisfy the following anti-commutation relations, $\{Q, Q^+\} = H, \{Q, Q\} = \{Q^+, Q^+\} = 0$,

$$\begin{aligned} &\{Q^+, Q^+\} = 0 \quad \Rightarrow \quad \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix} = 0, \\ &\{Q, Q\} = 0 \quad \Rightarrow \quad \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} = 0, \end{aligned} \tag{64}$$

and

$$\{Q, Q^{+}\} = H \quad \Rightarrow \quad \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \begin{pmatrix} 0 & A^{+} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & A^{+} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}$$
$$= \begin{pmatrix} A^{+}A & 0 \\ 0 & AA^{+} \end{pmatrix} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} = H. \tag{65}$$

6 Conclusion

In this paper, we introduced the generalized Kratzer potential (1) and wrote the corresponding Schrödinger equation. We connected this equation with Lagure and Jacobi equations and obtained the energy spectrum and wavefunction. Also we factorized the corresponding equation in terms of first order equations. We used these first order equations and discussed the generators algebra. Finally the partner for the Kratzer potential for angular and radial part is obtained by the supersymmetry approaches.

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